EEL 711/ELL 711 Major Test Semester I 2015-2016

Answer all questions (Marks: Q.1: 30, Q.2: 50) Full Marks: 80

1. Let $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_L \end{bmatrix}$, $L \geq 2$, be a $\mathcal{N}(\underline{0}_{L\times 1}, \underline{I}_L)$ random vector, and $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_L}{L}$.

The random vector $\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_L \end{bmatrix}$ is given by $Y_i \triangleq X_i - \beta \bar{X}$, $\quad i = 1, \ldots, L$, where β is a deterministic real constant. We can express $\underline{Y}$ as $\underline{Y} = \underline{B}\underline{X}$, where $\underline{B}$ is a deterministic $L \times L$ matrix. Let $V = \sqrt{\dfrac{\underline{X}^T\underline{Y}}{L}}$ and $U = \sqrt{\dfrac{\underline{Y}^T\underline{Y}}{L}}$.

(a) Find the two values of β for which V and U have the same distribution. [4]

(b) For the value of β for which both V and U are Nakagami distributed with second moment 1:

(i) Find the c.f. of V^2. [4]

(ii) Using the central limit theorem (CLT), find (in terms of L), the approximate c.f. of V^2 for large L. [4]

(iii) Find the c.d.f. of V^2 for $L = 6$. [2]

(iv) Calculate the probability $\Pr\left\{V > \frac{2}{\sqrt{3}}\right\}$ for $L = 6$ and for $L = 30$ (use the Q-function approximation wherever applicable). [2+2]

(c) For $L = 2, \beta = 1$ and $L = 2, \beta = 2$:

(i) Find the joint p.d.f. $f_{Y_1,Y_2}(y_1, y_2)$ and the c.f. of $Y_1^2 + Y_2^2$. [1+1+1+1]

(ii) Find the p.d.f. of $|Y_1 + Y_2|$. [2+2]

(d) Let the p.d.f. of $|Y_1 + Y_2|$ for $L = 2, \beta = 1$ be denoted by the function $g(\cdot)$. Let S_1, S_2, S_3, S_4 be i.i.d. random variables, with p.d.f.

$$f_{S_i}(x) = 0.4g(x-1) + 0.6g(x-2)\,, \quad i = 1, 2, 3, 4.$$

Calculate the probability $\Pr\left\{(S_1 + S_2 + S_3 + S_4)^2 < 40\right\}$. [4]

2. A WSS complex-valued circular Gaussian random process $Z(t)$ is expressed as

$$Z(t) = X(t) + \jmath Y(t)\,,$$

where $\jmath = \sqrt{-1}$ and $X(t)$ and $Y(t)$ are real-valued jointly WSS Gaussian random processes. The random process $Z(t)$ has mean $\mu_Z = \mu_X + \jmath\mu_Y$, with $\mu_X = \mathrm{Re}(\mu_Z) > 0$, $\mu_Y = \mathrm{Im}(\mu_Z) > 0$, autocorrelation function

$$R_Z(t_1, t_2) = \mathbf{E}\left[Z(t_1)Z^*(t_2)\right] = \left\{(256)^{-|t_1-t_2|}\cos\left(4\pi(t_1 - t_2)\right)\right\} + |\mu_Z|^2\,,$$

and pseudo-autocorrelation function

$$\tilde{R}_Z(t_1, t_2) = \mathbf{E}\left[Z(t_1)Z(t_2)\right] = 2 + \jmath\left(2\sqrt{3}\right).$$

$\tilde{K}_Z(t_1, t_2) = \mathbb{E}[(Z(t_1) - \mu_Z)(Z(t_2) - \mu_Z)]$
$= \tilde{R}_Z(t_1, t_2) - \mu_Z^2$.

(a) Find the following: [2+2+3+3]

(i) the mean μ_Z,

(ii) the cross-covariance function

$$K_{XY}(t_1, t_2) = \mathbf{E}\left[(X(t_1) - \mu_X)(Y(t_2) - \mu_Y)\right],$$

(iii) the autocovariance function

$$K_X(t_1, t_2) = \mathbf{E}\left[(X(t_1) - \mu_X)(X(t_2) - \mu_X)\right],$$

(iv) the autocorrelation function

$$R_Y(t_1, t_2) = \mathbf{E}\left[Y(t_1)Y(t_2)\right].$$

(b) Let

$$V = V_1 + jV_2 = Z(0) + 2Z\left(\frac{1}{4}\right) + 3Z\left(\frac{1}{2}\right),$$

where $V_1 = \mathrm{Re}(V)$, $V_2 = \mathrm{Im}(V)$.
(i) Find the joint p.d.f. $f_{V_1,V_2}(v_1, v_2)$. [6]
(ii) Find the c.f. of $V_1^2 - 3V_2^2$. [4]
(iii) Using Tchebycheff's Inequality, find an upper bound on the probability

$$\Pr\left\{\,|V - \mathbf{E}(V)| \geq 5\right\}$$

and also calculate the exact value of this probability. [3+3]

(c) Let $W(t) = X(t) - \alpha Y(t)$ be a zero-mean WSS random process. Find α, the p.s.d. $S_W(f)$ of $W(t)$, and the power of $W(t)$. [2+6+2]

(d) Define a random vector $\underline{W}$ as

$$\underline{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \triangleq \begin{bmatrix} X\left(\frac{1}{4}\right) - \alpha Y\left(\frac{1}{4}\right) \\ X\left(\frac{1}{2}\right) - \alpha Y\left(\frac{1}{2}\right) \end{bmatrix},$$

where α is as in (c). Let $\underline{B}$ be a 2×2 matrix such that $\underline{BW}$ has the p.d.f.

$$f_{\underline{BW}}(\underline{u}) = \frac{1}{2\pi} e^{-\frac{1}{2}\underline{u}^T\underline{u}}, \quad \underline{u} \in \mathcal{R}^2, \tag{1}$$

where $\mathcal{R}$ is the set of real numbers.
(i) Find a lower triangular matrix $\underline{B}$ that satisfies (1) and has positive diagonal elements. [6]
(ii) Calculate $\mathbf{E}\left[W_1 W_2^3\right]$, $\mathbf{E}\left[\underline{W}^T\underline{BW}\right]$, and $\mathbf{E}\left[\sqrt{\left(\underline{W}^T\underline{B}^T\underline{BW}\right)^3}\right]$. [2+2+4]